Exercise 1. Read and understand the proof of Ostrowski's theorem.

Exercise 2. Read and understand the proof that the different definitions of equivalence of absolute values are equivalent.

Exercise 3. Prove that $|\cdot|_p$ is not equivalent to neither $|\cdot|_q$ nor $|\cdot|_\infty$ for $p \neq q$ primes numbers.

Exercise 4. Let $n \geq 2$ and let $d \geq 2$ be a natural number such that there exists a prime p with $v_p(d)$ not a multiple of n. Show that the (positive real) n-th root of d is not a rational number.

Exercise 5. Let K be a field. Show that on the field of rational fractions $K(T)$ with coefficients in K, the valuation $-\deg$ and the T^{-1}-adic valuation are equal.

Exercise 6. Let $|\cdot|_\infty$ denote the standard absolute value on $\mathbb{R}$. Show that for $\alpha > 0$, $|\cdot|^\alpha$ is an absolute value if and only if $\alpha \leq 1$.

Exercise 7 (Valuations and absolute values)**.**
A totally ordered group $(\Gamma, +, \leq)$ is a group $(\Gamma, +)$ with a total order $\leq$ (meaning that for any $x, y \in \Gamma$, either $x \leq y$ or $y \leq x$) such that for all $x, y, z \in \Gamma$, $x \leq y$ implies $x + z \leq y + z$.

Let K be a field. A valuation on K is a map $v : K^\times \to \Gamma$ where Γ is a totally ordered group such that

(i) $v(xy) = v(x) + v(y)$;
(ii) if we extend v to a map of monoids $v : K \to \Gamma \cup \{\infty\}$ by $v(0) = \infty$ (where we declare $\infty > \gamma$ for all $\gamma \in \Gamma$), we have
$$v(x + y) \geq \min(v(x), v(y)).$$

(1) A valuation is said to be of height one if $\Gamma = (\mathbb{R}, \leq)$. Show that the map
$$\begin{array}{ccc} \left\{\begin{matrix}\text{height one} \\ \text{valuations on } K\end{matrix}\right\} & \longrightarrow & \left\{\begin{matrix}\text{non-archimedean} \\ \text{absolute values on } K\end{matrix}\right\} \\ v & \mapsto & \exp(-v(\cdot)) \end{array}$$
is a bijection.

A valuation ring is an integral domain R with fraction field K such that for any $x \in K$, either $x \in R$ or $x^{-1} \in R$.

(2) Show that a valuation ring is a local ring, by showing that its set of non-units is an ideal.
(3) Show that the relation $\overline{x} \leq \overline{y}$ if $yx^{-1} \in R$ define a total order on the quotient group $(\Gamma, +) := (K^\times / R^\times, \times)$, and that the quotient map $v : K^\times \to \Gamma$ is a valuation.
(4) Conversely, show that if $v : K^\times \to \Gamma$ is a valuation then the set $R := \{x \in K,\ v(x) \geq 0\}$ (where 0 denotes the identity element of $(\Gamma, +)$) is a valuation ring with maximal ideal $\mathfrak{m} := \{x \in K,\ v(x) > 0\}$.

Two valuations $v_1 : K^\times \to \Gamma_1$, $v_2 : K^\times \to \Gamma_2$ are called equivalent if there exists $v : K^\times \to \Gamma$ and order-preserving group embedding $\varphi_1 : \Gamma \to \Gamma_1$, $\varphi_2 : \Gamma \to \Gamma_2$ such that $v_1 = \varphi_1 \circ v$ and $v_2 = \varphi_2 \circ v$.

(5) Show that the above constructions induce a bijection

$$\left\{\begin{matrix}\text{valuation rings with} \\ \text{fraction field } K\end{matrix}\right\} \xrightarrow{\simeq} \left\{\begin{matrix}\text{equivalence classes of} \\ \text{valuations on } K\end{matrix}\right\}.$$

(6) Show that two height one valuations v_1 and v_2 on a field K are equivalent if and only if they define equivalent non-archimedean absolute values, if and only there exists $c \in \mathbb{R}_+^*$ such that $v_1(x) = cv_2(x)$ for all $x \in K$.

Exercise 8. The aim of this exercise is to study the radius of convergence of the exponential series on $\mathbb{Q}_p$ and its extensions.

(1) Let $N \in \mathbb{N}^*$ and let p be a prime number. Show that

$$v_p(p^N!) = \frac{p^N - 1}{p - 1}.$$

(2) Let $N \in \mathbb{N}^*$, let p be a prime number and let $a \in \{0, \dots, p-1\}$. Show that

$$v_p((ap^N)!) = a\frac{p^N - 1}{p - 1}.$$

(3) Let $n \in \mathbb{N}$. Show that

$$v_p(n!) = \frac{n - S_n}{p - 1}$$

where S_n denotes the sum of the digits of n in base p.

(4) Deduce a necessary and sufficient criterion of $|x|_p$ to have

$$\frac{x^n}{n!} \xrightarrow[n\to\infty]{} 0.$$

(5) Let K be a complete extension of $\mathbb{Q}_p$. Deduce for which $x \in K$ the series

$$\exp(x) := \sum_{n\geq 0} \frac{x^n}{n!}$$

converges.

Exercise 9. Show the following converse to the weak Tychonov theorem: if (X_i) is a countable family of compact (non-empty) metric spaces and X_0 (without loss of generality) is not compact, then $\prod_i X_i$ is not compact for the product topology.

Exercise 10. (*Hard*)

(1) Show that a metric space is compact if and only if it is complete and totally bounded.

(2) Deduce that a non-archimedean locally compact normed field with a non-trivial absolute value is complete and has a finite residue field.

(3) A non-trivial valuation on a field is called *discrete* if it is equivalent to a valuation with value group $\Gamma = (\mathbb{Z}, \leq)$. Show that a height one non-trivial valuation on a field K is discrete if and only if 0 is an isolated point in the value group $v(K^\times)$.
(4) Show that moreover, a locally compact non-archimedean normed field K with a non-trivial absolute value must satisfy that the associated valuation $v(\cdot) = \log(-|\cdot|)$ is discrete. (*Hint*: the equality case of the strong triangle inequality forces a convergent sequence to have the same absolute value as its limit after a certain rank.)
(5) Let K be a field with a non-trivial valuation v. Show that v is discrete if and only if the valuation ring $\mathcal{O}_K$ is principal, if and only if the maximal ideal $\mathfrak{m}$ is principal.
(6) Show the following converse to (2) and (4): if a normed field is complete, has finite residue field, and its valuation is discrete then it is locally compact.

Exercise 11. Do the analysis of chapter 3 but starting with the field $K(T)$ of rational fractions with coefficients in a field K, equipped with the T-adic absolute value $|\cdot|_T := \exp(-v_T(\cdot))$. Namely:

(1) What is its valuation ring, maximal ideal, and residue field ?
(2) Denote by L its completion and $\mathcal{O}_L$ the corresponding valuation ring. Identify $\mathcal{O}_L$ as the ring of formal power series $K[[T]]$ with coefficients in K. What is its maximal ideal ? What is its residue field ? More generally, describe its ideals and the corresponding quotient rings.
(3) Write $K[[T]]$ as a limit.
(4) When is $K[[T]]$ compact for the topology induced by the T-adic valuation ?
(5) Deduce from your answer above an example of a metric space that is complete but not locally compact.

Notice that in a way the p-adic integers behave like a power series in a formal variable p with coefficients in $\mathbb{F}_p$, except that $(p-1) \cdot p^0 + 1 \cdot p^0 = 1 \cdot p^1$ and other similar phenomena, so addition and multiplication is not performed term-by-term as in power series. However, see the following exercise:

Exercise 12 (Hard)**.** Construct an isomorphism of rings

$$\mathbb{Z}[[X]]/(X-p) \xrightarrow{\simeq} \mathbb{Z}_p$$

sending X to p. Bonus question: show that if we endow $\mathbb{Z}[[X]]$ with the X-adic topology, i.e. the topology induced by the metric induced by the X-adic valuation, and endow $\mathbb{Z}[[X]]/(X-p)$ with the quotient topology, then the above is a homeomorphism.

Exercise 13 (Not so relevant)**.** Read up on quaternions and show that the quaternions with the Euclidean norm is a "complete locally compact archimedean normed non-commutative field".