# BMST 2025: $p$-ADIC NUMBERS
# EXERCISE SHEET 3

The following exercise shows that many of the properties of the $p$-adic integers are shared with rings of power series with coefficients in a field.

**Exercise 1.** Do the analysis of chapter 3 but starting with the field $K(T)$ of rational fractions with coefficients in a field $K$, equipped with the $T$-adic absolute value $|\cdot|_T := \exp(-v_T(\cdot))$. Namely:

(1) What is its valuation ring, maximal ideal, and residue field ?
(2) Denote by $L$ its completion and $\mathcal{O}_L$ the corresponding valuation ring. Identify $\mathcal{O}_L$ as the ring of formal power series $K[[T]]$ with coefficients in $K$. What is its maximal ideal ? What is its residue field ? More generally, describe its ideals and the corresponding quotient rings.
(3) Write $K[[T]]$ as a limit.
(4) When is $K[[T]]$ compact for the topology induced by the $T$-adic valuation ?
(5) Deduce from your answer above an example of complete non-archimedean field, with a discrete absolute value, that is not locally compact.

In a way, the $p$-adic integers behave like power series in a formal variable $p$ with coefficients in $\mathbb{F}_p$, except that $(p-1)\cdot p^0 + 1\cdot p^0 = 1\cdot p^1$ and other similar phenomena, so addition and multiplication is not performed term-by-term as in power series.

In a non-archimedean field with a (non-trivial) discrete valuation, a *uniformizer* is a generator $\pi$ of the maximal ideal $\mathfrak{m}$, i.e. an element of maximal absolute value strictly less than 1.

**Exercise 2** (Canonical lifts)**.** Let $p$ be a prime and let $K$ be a non-archimedean normed field with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}$ and residue field $k$. We assume that $p \in \mathfrak{m}$.[1] Recall that for $1 \leq k \leq p-1$, the binomial coefficient $\binom{p}{k}$ is divisible by $p$.

(1) Show that for $x, y \in \mathcal{O}_K$, if $|x-y| < 1$ then for $t = \max(|p|, |x-y|) < 1$ we have$|x^{p^k} - y^{p^k}| \leq t^{k+1}$ for all $k \geq 1$.
(2) Reformulate this in terms of congruences in the case where the valuation is discrete.

We now assume that $K$ is complete with a discrete valuation and that the residue field $k$ of $K$ is perfect, that is such that any $x \in k$ has a (unique)

[1] This means that either $p = 0$ in $K$, or $K$ is of characteristic 0 and contains $\mathbb{Q}$ with its $p$-adic absolute value. Thus, there always exists such a $p$.

$p$-th root (i.e. $y \mapsto y^p$ is an automorphism of $k$, since $p = 0$ in $k$). We let $\pi \in \mathfrak{m}$ be a uniformizer.

(3) For $x \in k$, let $x^{p^{-k}}$ denote its unique $p^k$-th root. For each $k \in \mathbb{N}$, choose a lift $\widetilde{x^{p^{-k}}} \in \mathcal{O}_K$ of $x^{p^{-k}}$. Show that the expression
$$\tau(x) := \lim_{k \to \infty} \left( \widetilde{x^{p^{-k}}} \right)^{p^k}$$
is well-defined and independent of the choice of lifts.

(4) Deduce that the map $\tau : k \to \mathcal{O}_K$ is a multiplicative injection with image a complete set of representative of classes modulo $\mathfrak{m}$.

(5) Show that if $K$ is of characteristic $p$, $\tau$ is a field embedding of $k$ in $K$ with image in $\mathcal{O}_K$, and $\pi \circ \tau = \mathrm{id}$ where $\pi : \mathcal{O}_K \to k$ denotes the canonical projection.

(6) Show that every element of $\mathcal{O}_K$ has a unique representation in base $\pi$ with coefficients in the image of $\tau$ (*Hint*: think of how we proved that elements in $\mathbb{Q}_p$ have a base $p$ expansion).

(7) Deduce that if $K$ is a complete non-archimedean normed field of characteristic $p$ with a non-trivial discrete valuation and a perfect residue field, there are isomorphisms of normed rings $\mathcal{O}_K \simeq k[[X]]$ and normed fields $K \simeq k((X))$, where $k((X))$ has the $X$-adic valuation.